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INFLUENCE OF MODAL PARAMETERS UNCERTAINTIES ON THE ACTIVE CONTROL OF AN ELASTIC STRUCTURE

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Abstract. *The objective of this paper is to analyze the influence of uncertainties in the values of modal parameters on the active control of an elastic structure. Random variations are applied to the correct values of natural frequencies, damping ratios and modal shapes of the structure, and the resulting effects on the performance of the control system are studied. The structure is assumed to present proportional damping. The control is applied independently to a few modes, under a discrete-time scheme. The actuators are only linked to points within the structure, thus providing "internal control forces". Computer simulation results are presented and discussed.*

Key-words: *Active control, Vibrations, Elastic structure*

1. INTRODUCTION

The application of the independent modal-space control (IMSC), introduced by Meirovitch et al. (1983) and described below, implies the knowledge of some modal parameters of the structure under control. If these parameters are not correctly identified, it is expected that the performance of the control scheme will be degraded. The aim of this paper is to simulate some uncertainties in the modal parameters in order to quantify the resulting variations on the controlled response of the structure. The variations applied to the modal parameters are of the same order, in value, that the errors usually obtained in experimental modal analysis.

The following sections present the theory of IMSC, the elastic structure used in the computer simulations and the results of these simulations. The analysis of the results and the conclusions are the ending sections.

2. IMSC THEORY

The dynamic equation of a linear self-adjoint elastic structure, with time-invariant parameters, represented by its mass (**M**), damping (**C**) and stiffness (**K**) matrices, is given by:

$$\mathbf{M}\,\ddot{\mathbf{x}}(t) + \mathbf{C}\,\dot{\mathbf{x}}(t) + \mathbf{K}\,\mathbf{x}(t) = \mathbf{D}\,\mathbf{f}(t)\ , \qquad (1)$$

where **x**(t) is the n-dimensional vector of the coordinates used to describe the movement of the structure. $\dot{\mathbf{x}}(t)$ and $\ddot{\mathbf{x}}(t)$ are respectively the first and second time derivatives of $\mathbf{x}(t)$. Matrix **D,** with dimensions n × nf, distributes the nf forces contained in vector **f**(t) along the coordinates of the structure.

Considering that the damping is of the proportional type, matrices **M**, **C** and **K** may be diagonalized by matrix **Φ**, that presents as its columns the real eigenvectors obtained from the pair of matrices [**M**, **K**]. These eigenvectors are normalized with respect to the mass matrix, resulting the relationships:

$$\mathbf{\Phi}'\,\mathbf{M}\,\mathbf{\Phi} = \mathbf{I}, \quad \mathbf{\Phi}'\,\mathbf{K}\,\mathbf{\Phi} = \mathbf{\Lambda} \quad \text{and} \quad \mathbf{\Phi}'\,\mathbf{C}\,\mathbf{\Phi} = \mathbf{\Sigma}\ , \qquad (2)$$

where **I** is the identity matrix, **Λ** is a diagonal matrix with the values of the squared undamped natural frequencies of the modes ($\mathbf{\Lambda} = diag\{\omega_{\mathbf{n_r}}^2\}$) and **Σ** results from the diagonalization of the damping matrix ($\mathbf{\Sigma} = diag\{2\,\xi_{\mathbf{r}}\,\omega_{\mathbf{n_r}}\}$). $\xi_{\mathbf{r}}$ is the damping ratio of the r-th mode. The superscript ′ denotes the transpose of a vector or matrix.

Matrix **D** allows the same force to be applied at different points of the structure, when there are two or more non-zero values in one of its columns. So, there is the possibility of using "ïnternal forces" (Jordan and Arruda, 1991 and Jordan, 1993), that is, forces provided by a control actuator linking two points of the same structure.

The physical coordinates **x**(t) are now changed into a set of generalized coordinates $\boldsymbol{\eta}(t)$ by the following equation:

$$\mathbf{x}(t) = \mathbf{\Phi}\,\boldsymbol{\eta}(t)\,. \qquad (3)$$

Taking into account Eq. (2) it is possible to see that $\mathbf{\Phi}^{-1} = \mathbf{\Phi}'\,\mathbf{M}$, so that each generalized coordinate is given by:

$$\eta_r(t) = \boldsymbol{\phi}_{\mathbf{r}}'\,\mathbf{M}\,\mathbf{x}(t)\,, \qquad (4)$$

where $\boldsymbol{\phi}_{\mathbf{r}}$ is the eigenvector of the corresponding mode.

Introducing Eq. (3) into Eq. (1) and pre-multiplying the resulting one by $\mathbf{\Phi}'$, it yields:

$$\ddot{\boldsymbol{\eta}}(t) + \mathbf{\Sigma}\,\dot{\boldsymbol{\eta}}(t) + \mathbf{\Lambda}\,\boldsymbol{\eta}(t) = \mathbf{f_g}(t)\,, \qquad (5)$$

where the generalized forces $\mathbf{f_g}(t)$ are given by:

$$\mathbf{f_g}(t) = \mathbf{\Phi}'\,\mathbf{D}\,\mathbf{f}(t)\,. \qquad (6)$$

The modes are now uncoupled and, from Eq. (5), it is possible to write for each of them:

$$\ddot{\eta}_r(t) + 2\,\xi_r\,\omega_{nr}\dot{\eta}_r(t) + \omega_{nr}^2\,\eta_r(t) = f_{g_r}(t). \tag{7}$$

Equation (7) can be rewritten in the following matrix form:

$$\begin{bmatrix}\dot{\eta}_r(t)\\ \ddot{\eta}_r(t)\end{bmatrix} = \begin{bmatrix}0 & 1\\ -\omega_{nr}^2 & -2\,\xi_r\,\omega_{nr}\end{bmatrix}\begin{bmatrix}\eta_r(t)\\ \dot{\eta}_r(t)\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix} f_{g_r}(t). \tag{8}$$

The above equation may be presented in a reduced form as:

$$\dot{\mathbf{z}}_{\mathbf{r}}(t) = \mathbf{A}_{\mathbf{r}}\,\mathbf{z}_{\mathbf{r}}(t) + \mathbf{B}_{\mathbf{r}}\,f_{g_r}(t), \tag{9}$$

where $\mathbf{z'}_{\mathbf{r}}(t) = \left[\eta_r(t) \quad \dot{\eta}_r(t)\right]$ and the matrices $\mathbf{A_r}$ and $\mathbf{B_r}$ are defined by:

$$\mathbf{A}_{\mathbf{r}} = \begin{bmatrix}0 & 1\\ -\omega_{nr}^2 & -2\,\xi_r\,\omega_{nr}\end{bmatrix} \quad \text{and} \quad \mathbf{B}_{\mathbf{r}} = \begin{bmatrix}0\\ 1\end{bmatrix}. \tag{10}$$

If Eq. (9) is integrated over a time step Δt, considering that the force $f_{g_r}(t)$ keeps virtually constant along this short time interval, it yields:

$$\mathbf{z}_{\mathbf{r}}(k+1) = \overline{\boldsymbol{A}}_{\mathbf{r}}\,\mathbf{z}_{\mathbf{r}}(k) + \overline{\boldsymbol{B}}_{\mathbf{r}}\,f_{g_r}(k), \tag{11}$$

where

$$\overline{\boldsymbol{A}}_r = e^{\mathbf{A}_r\,\Delta t} \quad \text{and} \quad \overline{\boldsymbol{B}}_r = \mathbf{A}_r^{-1}\left(e^{\mathbf{A}_r\,\Delta t} - \mathbf{I}\right)\mathbf{B}_r, \tag{12}$$

and k and k+1 indicate two sequential time events Δt apart.

In order to apply the independent modal control, it is now necessary to define for each mode a functional J_r to be minimized. According to Meirovitch et al. (1983), the functional defined by the following equation presents two terms. The first one involves the potential and kinetic energies of the mode for the k-th time instant. The second one takes into account the control wasted energy. The result is:

$$J_r = \frac{1}{2}\sum_{k=1}^{\infty}\left[\mathbf{z}_{\mathbf{r}}'(k)\;\mathbf{Q}_{\mathbf{r}}\;\mathbf{z}_{\mathbf{r}}(k) + R_r\;f_{g_r}^2(k)\right], \tag{13}$$

where

$$\mathbf{Q}_{\mathbf{r}} = \begin{bmatrix}\omega_{nr}^2 & 0\\ 0 & 1\end{bmatrix}. \tag{14}$$

Equations (11) and (13) define, for each mode, a discrete time control problem with infinite time horizon. The solution of this problem (Kirk,1970) leads to a constant feedback matrix G_r for each mode, so that the corresponding generalized force $f_{g_r}(k)$ is given by:

$$f_{g_r}(k) = \mathbf{G}_r \ \mathbf{z}_r(k) = \begin{bmatrix} G_{r_1} & G_{r_2} \end{bmatrix} \mathbf{z}_r(k) . \tag{15}$$

Analysis of Eq. (6) shows that if the number of control forces (that is, the number of independent actuators) is equal to the number of modes under control, the matrix product $\mathbf{\Phi_{nf}}' \ \mathbf{D}$ (where $\mathbf{\Phi_{nf}}$ is the matrix containing just the eigenvectors of the controlled modes) furnishes a square matrix. If the actuators are correctly positioned, this square matrix does not become singular, and so it admits inversion. As a consequence, there is a unique relationship between the physical and the generalized forces, given by:

$$\mathbf{f}(k) = \left(\mathbf{\Phi'_{nf}} \ \mathbf{D}\right)^{-1} \mathbf{f_g}(k) \tag{16}$$

To avoid the product $\mathbf{\Phi_{nf}}' \ \mathbf{D}$ to be a singular matrix it is necessary, for instance, that no one vector that represents a spatial force pattern (a column of **D**) is orthogonal to all the eigenvectors of the controlled modes (the columns of $\mathbf{\Phi_{nf}}$). The application of a single force at a point that is simultaneously a nodal point for all the controlled modes would fit this condition, but it would be very difficult to obtain in practice. Another possibility of getting a singular matrix would be the use of the same spatial force pattern for two independent forces (two identical columns of **D**), what obviously does not enhance the controllability of the structure and is consequently avoided.

It is possible to calculate the global response of the system by an equation similar to Eq. (11), obtained from Eq. (1), written in the form of a difference equation as:

$$\mathbf{w}(k+1) = \overline{\boldsymbol{A}} \ \mathbf{w}(k) + \overline{\boldsymbol{B}} \ \mathbf{f}(k) \tag{17}$$

where $\mathbf{w}'(k) = \begin{bmatrix}\mathbf{x}'(k) & \dot{\mathbf{x}}'(k)\end{bmatrix}$ and the matrices $\overline{\boldsymbol{A}}$ and $\overline{\boldsymbol{B}}$ are given by:

$$\overline{\boldsymbol{A}} = e^{\mathbf{A}\,\Delta t} \quad \text{and} \quad \overline{\boldsymbol{B}} = \mathbf{A}^{-1}\left(e^{\mathbf{A}\ \Delta t} - \mathbf{I}\right)\mathbf{B} , \tag{18}$$

and where the matrices **A** and **B** are defined by:

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C} \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} \mathbf{0} \\ \mathbf{M}^{-1}\mathbf{D} \end{bmatrix} . \tag{19}$$

The application of Eq. (17) depends on the previous determination of the generalized coordinates (Eq. (4) and its time derivative). The generalized forces are then obtained by Eq. (15), and finally the physical forces from Eq. (16).

3. TEST STRUCTURE

The structure used in the simulations is shown in Fig. 1. It is composed by welded steel tube elements, with total height of 1.5 m (five cells of 0.3 m each). Other dimensions are 0.4

and 0.6 m, respectively in the x and y directions. The tube elements have external and internal diameters of 13.1 and 10.5 mm. The geometrical properties of the cross sections are: area $A = 4.82 \times 10^{-5}$ mm^2, moment of inertia $I = 8.49 \times 10^{-10}$ mm^4 and polar moment of inertia $J = 1.70 \times 10^{-9}$ mm^4. The admitted material properties have the values: density $\rho = 7.8 \times 10^3$ kg/m^3, Young modulus $E = 2.07 \times 10^{11}$ N/m^2 and shear modulus of elasticity $G = 0.79 \times 10^{11}$ N/m^2.

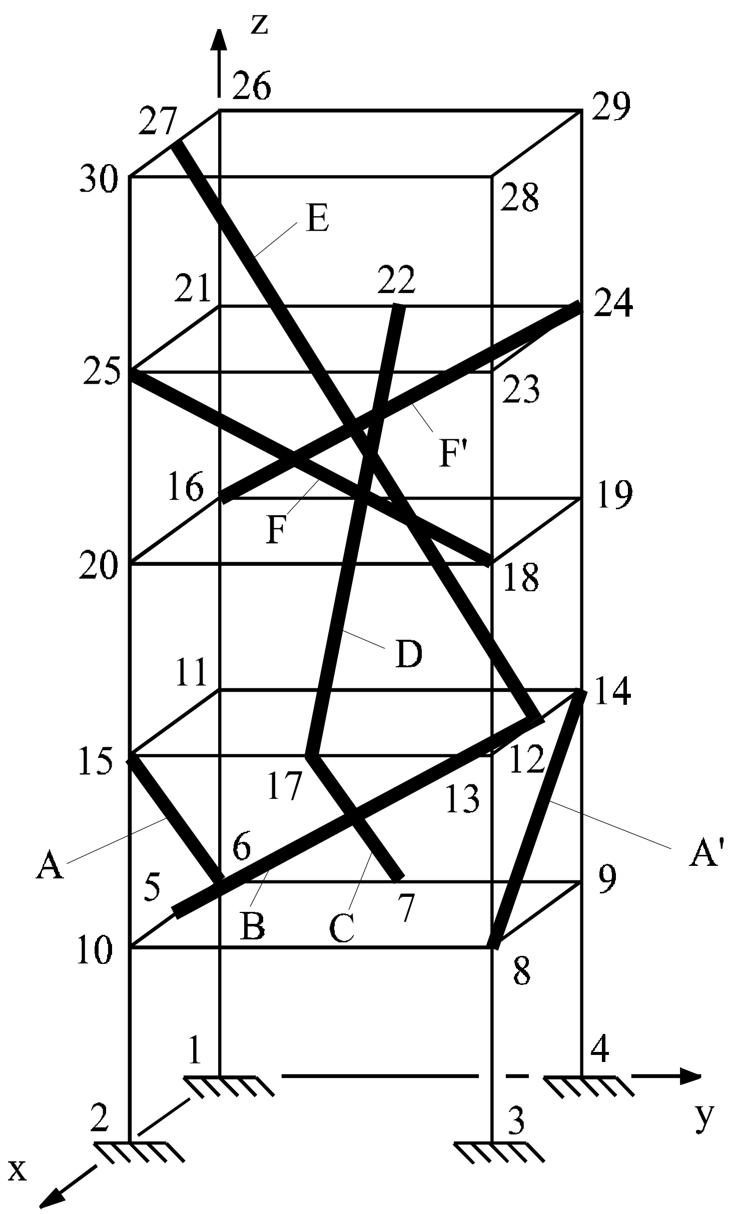

Figure 1 – Test structure

Figure 1 also shows the numbers that identify the limiting nodes of the 46 elements of the structure. These nodes are junction points and points where the actuators are connected to. There are 26 free nodes, with 6 degrees of freedom each, giving the total number of 156 degrees of freedom. Mass and stiffness matrices were obtained with the help of a finite element program based on ISMIS program (see Becker & Craig, 1974). With these matrices at hand, the corresponding eigenvalues and eigenvectors were calculated using the subroutine RITZIT (Wilkinson & Reinsch, 1971). The first six natural frequencies of the structure (derived from the eigenvalues) are presented in Tab. 1, with a description of the way the structure behaves when vibrating under their influence.

Table 1. Lowest natural frequencies of the structure

Order Number	Natural Frequencies [Hz]	Mode of Vibration
1	12.24	1st bending mode in y direction
2	13.87	1st bending mode in x direction
3	17.42	1st torsion mode
4	39.89	2nd bending mode in y direction
5	44.09	2nd bending mode in x direction
6	54.76	2nd torsion mode

The damping of the structure was taken as proportional, according to the relationship: $\mathbf{C} = \alpha\, \mathbf{M} + \beta\, \mathbf{K}$, with $\alpha = 0.7694$ and $\beta = 0$. The result was a damping ratio $\xi_1 = 0.005$ for the first mode and decreasing ratios for the following ones.

It is finally shown in Fig 1. the action lines of the actuators as the darker lines in the drawing. Since it was desired to control the six first modes of the structure (with the natural frequencies presented in Tab. 1.), six independent physical forces were necessary. The actuators are identified by the capital letters A to F'. It is important to notice that actuators A-A' apply identical physical control forces; the same happening to F-F'.

The random uncertainties were applied to the matrix of the eigenvectors ($\mathbf{\Phi}_{\mathbf{nf}}$) and to the values presented by the diagonal matrices obtained from the stiffness and damping matrices ($\mathbf{\Lambda}$ and $\mathbf{\Sigma}$). The values that define the eigenvectors have received deviations of up to 5 %. These deviations were applied independently to all the components of the eigenvectors, what means that the whole eigenvectors were not only subjected to random scaling factors. Maximum admitted errors at the estimation of the natural frequencies ($\boldsymbol{\omega}_{n_r}$) were taken as 1 %, while for the product $2\, \xi_r\, \omega_{n_r}$ the maximum uncertainties were of 20 %. These limits of deviations are supposed to represent commonly achieved errors in experimental modal analyses. The application of the random deviations upon the original matrices gave rise to thirty new matrices of each type ($\mathbf{\Phi}_{\mathbf{nf}}$, $\mathbf{\Lambda}$ and $\mathbf{\Sigma}$).

4. RESULTS FROM COMPUTATIONAL SIMULATIONS

The control theory was applied to the test structure, with the objective of controlling the modes with the lowest six natural frequencies. Initial conditions in terms of static deflections were applied to the structures. The structure was then abruptly released from this deformed shape to begin its vibration patterns. It is obvious then that no velocity initial conditions were considered. The deformed initial shape was calculated by the product $\mathbf{K}^{-1}\, \mathbf{f_e}$, where the static force vector $\mathbf{f_e}$ considered four 100 N forces applied at: node 14 in the direction +x, node 15 in –x, node 29 in –y and node 30 in +x. The chosen positions and directions of these forces were supposed to enhance the contributions of the six modes to the global response

Figure 2 shows two responses of the structure at node 26, in the y direction. The difference between the free and the controlled responses is clear. When calculating the controlled response it was used the values $R_r = 0.001$ (equal weighting factors for all controlled modes). Both responses were generated with the time increment $\Delta t = 0.001$s and a number of points equal to 250, giving a total time interval of approximately 0.25 s. This total time was sufficient to cover about three periods of the lowest natural frequency.

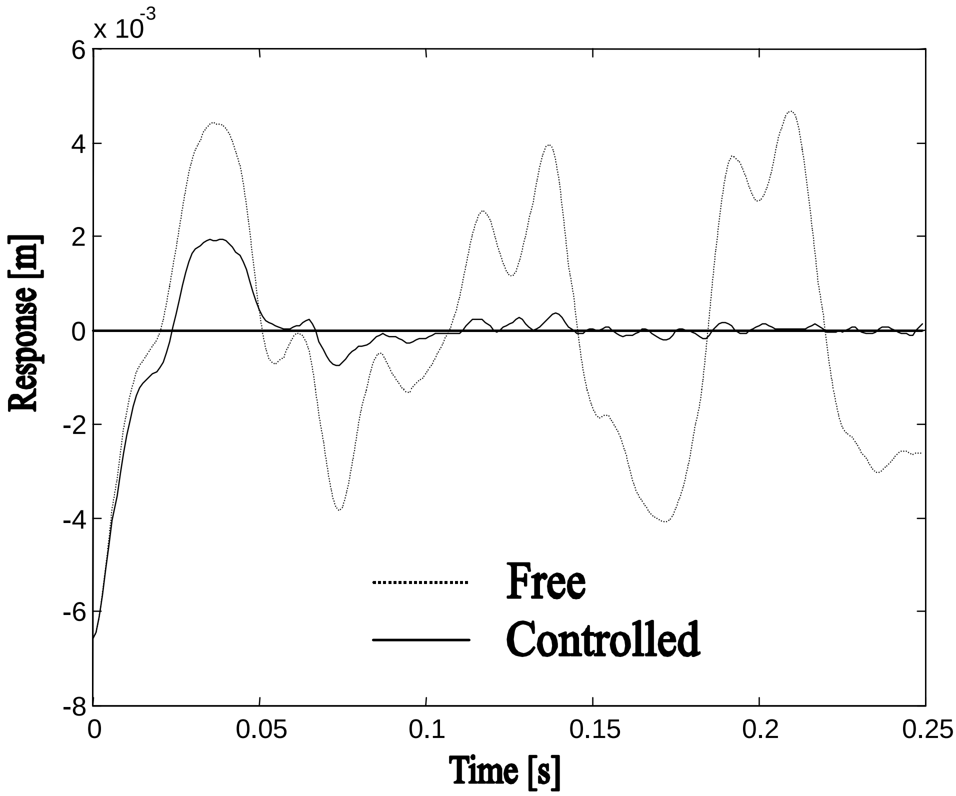

Figure 2 – Free and controlled responses at node 26 in the y direction

It is now performed the energy analysis of the response of the structure at the beginning and at the end of the 0.25 s period. The total energy $E_t(k)$, at instant k, is given by:

$$E_t(k) = \frac{1}{2}\left[\mathbf{x}'(k)\,\mathbf{K}\,\mathbf{x}(k) + \dot{\mathbf{x}}'(k)\,\mathbf{M}\,\dot{\mathbf{x}}(k)\right], \tag{20}$$

while the partial energy $E_{nf}(k)$, also at instant k, is given by (see Eq. (13)):

$$E_{nf}(k) = \frac{1}{2}\sum_{\mathbf{r}=1}^{\mathbf{nf}} \mathbf{z}_{\mathbf{r}}'(k)\;\mathbf{Q}_{\mathbf{r}}\;\mathbf{z}_{\mathbf{r}}(k)\ , \tag{21}$$

where nf is the number of controlled modes.

When the system is free, the total energy of the system decays from 0.592 to 0.489 Nm over this period. The initial energy of the six modes to be controlled, calculated according to the first term of Eq. (13), is equal in both responses (free and controlled): 0.574 Nm. At the end of the controlled response the total energy presented by the structure is 0.0198 Nm, and the energy possessed by these six modes is very low: 8.98×10^{-6} Nm. It is clear that the control of the targeted modes was very effective, although some spillover over other modes has occurred.

The thirty sets of modified matrices were then used to recalculate the controlled responses of the structure. The obtained mean value of the total energies at the end of the responses was of 0.0205 Nm (against 0.0198 Nm with the original system) with a standard deviation of 0.00122 Nm. The final energies of the six controlled modes have presented a mean value of 1.000×10^{-5} Nm (against 8.98×10^{-6} Nm with the original system) and a standard deviation of 0.196×10^{-5} Nm.

Remembering that the gain matrix of each controlled mode is composed by two elements (see Eq. (15)), it was decided to investigate the behavior of these values. Table 2 presents the mean values and the standard deviations of G_{r_1} and G_{r_2} for the six controlled modes.

Table 2. Analysis of the values of G_{r_1} and G_{r_2} for the controlled modes

	G_{r_1}			G_{r_2}		
Mode	Original Value	Mean Value	Standard Deviation	Original Value	Mean Value	Standard Deviation
1	-342.8	-343.2	0.897	-42.77	-42.76	0.0682
2	-310.3	-310.2	1.606	-42.85	-42,84	0.0665
3	-220.3	-220.3	2.060	-42.95	-42.96	0.0717
4	873.6	877.4	15.17	-42.84	-42.84	0.0719
5	1174.	1170.	17.24	-42.77	-42.75	0.0573
6	2071.	2066.	27.01	-42.56	-42.55	0.0639

5. ANALYSIS OF THE RESULTS

The mean value of the total energies at the end of the controlled responses (using the thirty sets of modified matrices) is only 3.54 % above the original value and the standard deviation is equal to 5.95 % of the mean value. With relation to the energies presented only by the six controlled modes, similar calculations reveal respectively the values of 11.4 % and 19.6 %.

For the values of G_{r_1} and G_{r_2}, the maximum differences between the original values and the mean values of the thirty samples is respectively 0.43 and 0.05 %. Again for G_{r_1} and G_{r_2}, the standard deviations have reached respectively the maximum values of 1.47 and 0.17 % of the corresponding mean values.

6. CONCLUSIONS

The influence of the deviations applied to the modal parameters on the resulting values G_{r_1} and G_{r_2} is very low, as can be seen in previous section. This means that the generalized forces and, as a consequence, the physical forces were not greatly affected by the changes. These little effects are cumulative when the response of the system is calculated, so that the energy of the six controlled modes presents some measurable differences at the end of the response period (there is a 11.4 % difference between the mean value of the samples and the original value, with a standard deviation that reaches almost 20 % of the mean value).

Fortunately, at the end of the considered period the energy of the six controlled modes is very low in comparison to the total energy presented by the structure, so that the observed change on the energy of the controlled modes is irrelevant.

The ending conclusion is that the expected uncertainties observed in the modal parameters obtained by experimental tests has a little influence on the performance of the control system, even when it is used the IMSC theory, essentially based on modal approach. Little changes at the modal feedback gain matrices were obtained, and they do not really modify the values of the physical control forces. So, the effective action of the control upon the target modes makes them to damp out very quickly, and the observed degradation on its performance is not really important.

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